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On-shell description of unsteady flames

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The problem of a non-perturbative description of unsteady premixed flames with arbitrary gas expansion is addressed in the two-dimensional case. Considering the flame as a surface of discontinuity with arbitrary local burning rate and gas velocity jumps, we show that the flame-front dynamics can be determined without having to solve the flow equations in the bulk. On the basis of the Thomson circulation theorem, an implicit integral representation of the downstream gas velocity is constructed. It is then simplified by a successive stripping of the potential contributions to obtain an explicit expression for the rotational component near the flame front. We prove that the unknown potential component is left bounded and divergence-free by this procedure, and hence can be eliminated using the dispersion relation for its on-shell value (i.e. the value along the flame front). The resulting system of integro-differential equations relates the on-shell fresh-gas velocity and the front position. As limiting cases, these equations contain all the theoretical results on flame dynamics established so far, including the linear equation describing the Darrieus–Landau instability of planar flames, and the nonlinear Sivashinsky–Clavin equation for flames with weak gas expansion.

1. Introduction

Although the elementary physical mechanisms underlying flame propagation are now well-understood, a global mathematical description of the process remains extremely difficult. The reasons for this – which flames share with condensation discontinuities in supersaturated vapours (Landau & Lifschitz 1987), ablation fronts driven by lasers or X-rays (Clavin, Masse & Williams 2005), thermonuclear waves in type-Ia supernovæ (Hillebrandt & Niemeyer 2000) or rapid decomposition of explosive liquids (Landau & Lifschitz 1987) – can be summarized briefly: all involve propagating free boundaries crossed by non-zero mass fluxes and separating subsonic flows of markedly different densities. In view of this, it is not surprising that, following the identification of flames as self-propagating slow deflagrations (as opposed to detonations) in the 1880s (Mallard & Le Chatelier 1881), they had to wait some 50 years before their theoretical study began at the simplest level of a linear stability analysis of planar flames (Darrieus 1938; Landau 1944). It took another 40 years for the first consistent account of nonlinear effects to appear. It was shown by Sivashinsky (1977) how these effects can be described in the case where the fresh-to-burnt gas density ratio, θ, is close to unity. The latter condition is a limitation of principle for a *perturbative* treatment of nonlinear saturation phenomena. Indeed, weak nonlinearity expansions are self-contradictory for the case of steady flames with a finite $(\theta - 1)$ (Kazakov & Liberman 2002). Nevertheless, despite the fact that the practically relevant values of θ are 5–8, the approximation of small $(\theta - 1)$ has so far been the only theoretical method available to handle flame-front dynamics. (We do not consider models, such as Frankel's potential-flow model (Frankel 1990) that cannot be deductively obtained from the fundamental equations. Frankel's model indeed assumes both finite values of $(\theta - 1)$ and zero vorticity production in the flame front; these two assumptions are only consistent in the lowest order of an expansion in small $(\theta - 1)$.)

One of the essential difficulties encountered in any analytic treatment of flames is the virtual impossibility of solving the flow equations governing the dynamics of the burnt gases. This is an exceedingly complicated problem which requires finding solutions to a system of nonlinear partial differential equations in the regions upstream and downstream of the flame front, chosen so as to satisfy a number of jump conditions expressing the conservation of mass, energy and momentum across the moving front. The dynamics of the front itself is determined by an evolution equation describing the local fresh-gas consumption rate as a functional of the fresh-gas velocity distribution along the front and the shape of the front (Markstein 1951). Even if the gas flow is potential upstream, as is the case for flames propagating in an initially quiescent fluid, this property is lost in the downstream region because vorticity is generated by the curved flame front, so that the problem of solving the flow equations is faced in its full generality.

Struggling with this problem is indeed unavoidable if one is interested in the explicit structure of the burnt-gas flow. However, in practice, the main concern is usually the evolution of the position and shape of the flame front. This limitation of the problem naturally raises the following dilemma (see Kazakov 2005*a*, *b*). On the one hand, the deflagration is an essentially non-local and nonlinear process with all the complications mentioned above; on the other hand, this non-locality itself is determined by the configuration of the flame front and the gas velocity distribution along it, which plays the role of boundary conditions for the flow equations and thus controls the bulk flow. Under such circumstances, is it really necessary to explicitly know the flow structure in the entire downstream region in order to describe the evolution of the front in a closed form, i.e. in a form involving only quantities defined on the flame front?

For steady configurations, this question was settled in the negative by Kazakov (2005*a*, *b*). More precisely, it was shown that the only piece of information about the downstream gas flow that is actually necessary to derive an equation for the flame-front position is the value of the rotational component of burnt gas along the front or, to borrow the terminology used by Kazakov (2005*b*), the *on-shell value* of this component. The remaining unknown potential component of the gas velocity is eventually excluded using a 'dispersion relation' for its on-shell values (expressing analyticity of this component in the downstream region), thus providing a description of flame shapes in a form that is closed in the above-mentioned sense. The purpose of the present paper is to generalize this construction to the case of unsteady flame propagation. Although the procedure is essentially the same as in the steady case, a subtle point is worth being emphasized. According to Kazakov (2005*a*, *b*), the spatial derivative of the rotational component along the flame front is local, i.e. its value at a given point is a function of the on-shell fresh-gas velocity, its derivatives, and the front shape at the same point. In view of this, one might expect that the generalization to the unsteady case is purely kinematic: namely, that it would be just a question of rewriting the steady equation in terms of the gas velocities relative to the local front velocity. We will see, however, that this is not so because of a peculiar – yet unavoidable – spatial non-locality of the rotational component, which appears naturally in the unsteady case to account for the effect of the flame history on its current evolution. Because of this complication, it is necessary to be more careful with the spatial integrations involved in the definition of the rotational component. To define the improper integrals along the flame front, we use an intermediate regularization. More specifically, we introduce an exponential damping of the contributions originating from remote parts of the front. This regularization is eventually removed through analytic continuation to the case of zero damping.

The paper is organized as follows. After formulating a spatially periodic version of the problem (§ 2), we construct an implicit integral of the flow equations downstream on the basis of the Thomson circulation theorem, which expresses gas velocity in terms of its boundary values and the vorticity distribution behind the front. This is done in § 3. The integral representation is then used in § 4 to obtain an expression for the rotational component of the gas velocity near the front, which is accomplished by successive stripping of the potential contributions to the burnt-gas velocity. We prove that the unknown potential component is left bounded and divergence-free by this procedure. Hence, it can be eliminated using the dispersion relation for its on-shell value. This leads to the main integro-differential equation obtained in § 5. This equation relates the on-shell value of the fresh-gas velocity and the flame front position, and together with the evolution equation constitutes the closed system for these quantities. Finally, it is verified in § 6 that the equation thus derived contains all the known theoretical results on flame dynamics as simple limiting cases, namely, the linear equation describing Darrieus–Landau instability of planar flames (Darrieus 1938; Landau 1944), including the case where the flame propagates in a time-dependent gravitational field (Markstein 1964), the nonlinear Sivashinsky–Clavin equation for flames with weak gas expansion (Sivashinsky 1977; Sivashinsky & Clavin 1987), and the stationary equation derived by Kazakov (2005*b*). The Appendix contains a consistency check for the results obtained in § 4.

2. Formulation

Consider a flame propagating in an initially quiescent uniform premixed ideal fluid. Our analysis below relies substantially on the well-known Thomson theorem stating that circulation of the gas velocity over any closed material contour drawn in an ideal fluid is conserved as it is convected. This statement takes on a particularly simple form in the case of two-dimensional incompressible flows, since not only is the circulation itself conserved but also the value of vorticity carried by any fluid element. Since space dimensionality is not particularly important in the *formulation* of the dilemma mentioned in the introduction, in the following we will limit our discussion to the simpler two-dimensional case, for which the necessary mathematical tools are available. We will further limit our analysis to flames propagating in a channel of constant width b. Let the Cartesian coordinates (x, y) be chosen so that the y-axis is parallel to the tube walls, $y = -\infty$ being in the fresh gas. These coordinates will be measured in units of the channel width, while fluid velocity, $\boldsymbol{v} = (v_1, v_2)$, is in units of the velocity of a plane flame front relative to the fresh gas. It will sometimes be useful to denote the Cartesian components of $\boldsymbol{v}$ by (w, u). Finally, the fluid density will be normalized by the fresh-gas density, $\theta > 1$ then denotes its ratio to that of the burnt gas.

It will be appropriate for our purposes to reformulate the problem under consideration as a problem of propagation of an unbounded spatially-periodic flame. Assuming that the channel walls are ideal, and given a flame configuration described by the functions $f(x,t)$, $\boldsymbol{v}(x,y,t)$, $x \in [0,+1]$, where $f(x,t)$ denotes the flame-front position at instant t, using the boundary conditions $\partial f/\partial x = 0$, $w = 0$ for $x = 0, 1$, we carry this configuration over to the domain $x \in [-1,0]$ according to

$$f(x,t) = f(-x,t), \quad w(x,y,t) = -w(-x,y,t), \quad u(x,y,t) = u(-x,y,t), \tag{2.1}$$

and then periodically continue it along the whole x-axis. Note that imposing the boundary condition $\partial f/\partial x$ for $x = 0, 1$ excludes the possibility of forming a stagnation zone near the end points of the flame front (see Zel'dovich *et al.* 1980 for details). We also assume that the flame is stable with respect to short wavelength perturbations, i.e. that there is a short wavelength cutoff, λ_c. This cutoff ensures smoothness of the functions under consideration. In particular, it prevents the development of singularities, such as edge points, which would otherwise occur (Zel'dovich *et al.* 1980), leading to discontinuities in the values of the flow variables or their derivatives. Since λ_c often significantly exceeds the actual thickness of the flame preheat zone (Searby & Rochwerger 1991), it has yet another virtue: the Reynolds number based on λ_c and the fresh-gas properties is typically over $\sim 10^2$, and hence is even larger when based on the width ($> \lambda_c$ or $\gg \lambda_c$) of the channel where the flame studied below is meant to propagate. It then makes sense to model the flame as a surface (or a line in two-dimensions) equipped with a local λ_c-dependent propagation law, embedded in an ideal fluid flow. We shall return to this issue in the final section of the paper, mentioning here that it is known from direct numerical simulations (Bychkov & Liberman 2000) that viscosity has a negligible influence on the shape and speed of steady curved flames.

In our formulation, flow velocity obeys the following equations in the bulk:

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{2.2}$$

$$\frac{\partial \sigma}{\partial t} + v_i \frac{\partial \sigma}{\partial x_i} = 0, \tag{2.3}$$

where $(x_1, x_2) = (x, y)$, and

$$\sigma = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \tag{2.4}$$

is vorticity. Summation over repeated indices is implicit. Equation (2.2) expresses the continuity of incompressible flows, while (2.3) constitutes the Thomson theorem.

3. Integral representation of the flow equations

It is not difficult to see that $\boldsymbol{v}$ satisfying (2.2), (2.4) can be written in the following integral form (cf. derivation of equation (9) in Kazakov 2005*b*)

$$v_i = \varepsilon_{ik}\partial_k \int_\Lambda \mathrm{d}l_l \; \varepsilon_{lm} v_m \frac{\ln r}{2\pi} - \partial_i \int_\Lambda \mathrm{d}l_k \; v_k \frac{\ln r}{2\pi} - \varepsilon_{ik}\partial_k \int_\Sigma \mathrm{d}s \; \frac{\ln r}{2\pi}\sigma, \tag{3.1}$$

where $\varepsilon_{ik} = -\varepsilon_{ki}$, $\varepsilon_{12} = +1$, $\partial_i = \partial/\partial x_i$. Σ and Λ denote any part of the downstream region and its boundary, respectively. r is the distance between an infinitesimal fluid element $\mathrm{d}s$ at the point $(\tilde{x}, \tilde{y})$ and the point of observation $\boldsymbol{x} = (x, y) \in \Sigma$, $r^2 = (x_i - \tilde{x}_i)^2$, and $\mathrm{d}l_i$ is the line element normal to Λ and directed outwards from

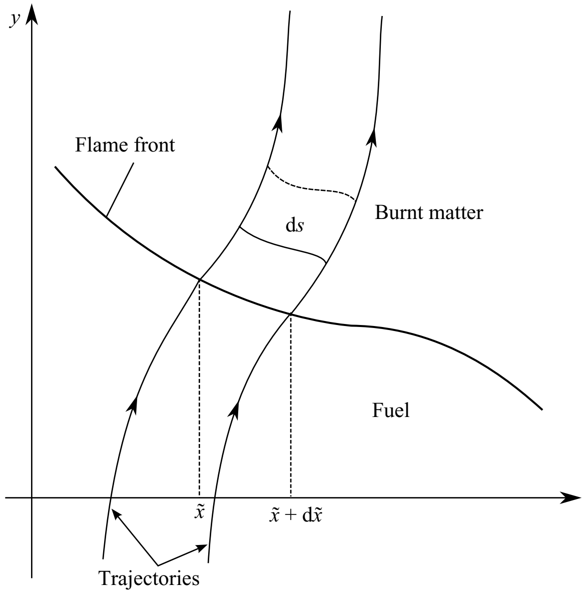

FIGURE 1. Elementary decomposition of the flow downstream used in the derivation of the expression (3.3).

Σ. In order to evaluate the divergence of the right-hand side of (3.1), we use the relations $\partial_i \varepsilon_{ik} \partial_k \equiv 0$,

$$\partial_k^2 \ln r = 2\pi\delta(x - \tilde{x})\delta(y - \tilde{y}), \tag{3.2}$$

where $\delta(x)$ is the Dirac delta-function. It follows that $\partial_i v_i = 0$ for any point $\boldsymbol{x}$ inside Σ. Similarly, evaluation of the curl of (3.1) with the help of (3.2) and $\varepsilon_{ik}\varepsilon_{im} = \delta_{km}$ gives the identity $\varepsilon_{ik}\partial_i v_k = \sigma$. We will now use (2.3) to rewrite the last term in (3.1) as an integral over fluid particle trajectories.

To this end, we specify that Σ is spanned by fluid elements that crossed the flame front between two points with fixed abscissas $\tilde{x}_1 = -A$ and $\tilde{x}_2 = +A$ during the time interval $[-T, t]$, where t is the given time instant at which the flow is observed (see figure 1). Here A and T are positive parameters, eventually tending to infinity, so that Σ will go on to fill the whole downstream region: $\Sigma = \{\tilde{x}, \tilde{y} : \tilde{y} > f(\tilde{x}, t)\}$. But for the time being, we keep A and T finite. The improper integral over an infinite Σ will be defined in §4.1. By virtue of the conservation of vorticity, we then have:

$$\int_\Sigma \mathrm{d}s\ \sigma \ln r = \int_{-A}^{+A} \mathrm{d}\tilde{x} \int_{-T}^{t} \mathrm{d}\tau N(\tilde{x}, \tau)\bar{v}_+^n(\tilde{x}, \tau)\sigma_+(\tilde{x}, \tau) \ln\{[x - X(\tilde{x}, t, \tau)]^2 + [y - Y(\tilde{x}, t, \tau)]^2\}^{1/2}, \tag{3.3}$$

where $N = \sqrt{1 + (\partial f/\partial x)^2}$, $\bar{v}_+^n = \bar{v}_{i+} n_i$ is the normal burnt-gas velocity relative to the flame front,

$$\bar{\boldsymbol{v}}_+ = (w_+, \bar{u}_+), \quad \bar{u}_+(x, t) \equiv u_+(x, t) - \frac{\partial f(x, t)}{\partial t},$$

n_i is the unit vector normal to the front (pointing towards the burnt gas), and $(X(\tilde{x}, t, \tau), Y(\tilde{x}, t, \tau))$ is the current position of a fluid element that crossed the point $(\tilde{x}, f(\tilde{x}, \tau))$ on the flame front at τ. In (3.3) we have used the fact that the 'volume' ds of this element is conserved because of flow incompressibility, and can hence be written as $\mathrm{d}\tilde{x}\mathrm{d}\tau N(\tilde{x},\tau)\bar{v}_+^n(\tilde{x},\tau)$. Changing the variable of integration, $\tau \to t-\tau$, in expression (3.3), and substituting it into (3.1) gives

$$v_i = \varepsilon_{ik}\partial_k \int_\Lambda \mathrm{d}l_l\ \varepsilon_{lm}v_m \frac{\ln r}{2\pi} - \partial_i \int_\Lambda \mathrm{d}l_k\ v_k \frac{\ln r}{2\pi} - \frac{\varepsilon_{ik}}{2}\partial_k \int_{-A}^{+A} \mathrm{d}\tilde{x} K(x,y,\tilde{x},t), \quad (3.4)$$

where the integral kernel K is defined by

$$K(x,y,\tilde{x},t) = \frac{1}{\pi}\int_0^{T+t} \mathrm{d}\tau M(\tilde{x},t-\tau) \times \ln\{[x - X(\tilde{x},t,t-\tau)]^2 + [y - Y(\tilde{x},t,t-\tau)]^2\}^{1/2}, \quad (3.5)$$

$$M(\tilde{x},\tau) \equiv N(\tilde{x},t)\bar{v}_+^n(\tilde{x},t)\sigma_+(\tilde{x},t). \quad (3.6)$$

4. Near-the-front structure of the rotational component

The integral representation (3.4) of the downstream flow velocity is to be used below to obtain an expression for its rotational component near the flame front. More precisely, we will define a rotational component v_i^v in a way that will allow an explicit expression for its on-shell value in terms of the on-shell gas velocity v_{i+}. For this purpose, we decompose the velocity field as

$$v_i = v_i^p + v_i^v, \quad i = 1, 2,$$

where v_i^p is a potential component satisfying the following requirements ($\mathrm{D}\boldsymbol{v}^p$ denotes any first-order spatial derivative of $\boldsymbol{v}^p$):

(a) $\operatorname{div} \mathrm{D}\boldsymbol{v}^p = 0$,

(b) $\operatorname{rot} \mathrm{D}\boldsymbol{v}^p = 0$,

(c) $\mathrm{D}\boldsymbol{v}^p$ is bounded, in the sense that it remains finite in the limit $A, T \to +\infty$, i.e. for an infinitely expanding Σ region.

The on-shell expression for the vorticity contribution will be obtained by gradually removing potential fields fulfilling conditions (a)–(c) above from (3.4). Although this derivation closely follows that of Kazakov (2005*b*), we give it here in detail in order to make clear the point where the non-stationarity of the problem comes into play.

Let the equality of two functions $\varphi_1(x,y)$, $\varphi_2(x,y)$ up to a field satisfying (a)–(c) be denoted by $\varphi_1 \overset{\circ}{=} \varphi_2$. As we saw in the preceding section, the first two terms on the right-hand side of (3.4) have vanishing curl and divergence, and hence also satisfy requirements (a) and (b). Furthermore, a simple power counting shows that (c) is likewise met. Indeed, consider the part $\Lambda\backslash F$ of the contour Λ, where F denotes the flame front. Representing this part as a semicircle with radius $R \to \infty$, we note that $\mathrm{D}^2 \ln r = O(1/R^2)$ for any given $\boldsymbol{x} \in \Sigma$ (D^2 denotes any second spatial derivative). Taking also into account that $\boldsymbol{v} = O(1)$, $\mathrm{d}l = R\,\mathrm{d}\phi$, where $\phi \in (0,\pi)$ is the angular coordinate of the point $\tilde{\boldsymbol{x}}$ on the semicircle, one sees that, after spatial differentiation, the two integrals over $\Lambda\backslash F$ on the right-hand side of (3.4) vanish in the limit $R \to \infty$. Similar considerations show that the same integrals over F are convergent, thus proving that $\mathrm{D}\boldsymbol{v}^p$ remains bounded downstream in the limit $A, T \to \infty$. So we can write

$$v_i \overset{\circ}{=} -\frac{\varepsilon_{ik}}{2}\partial_k \int_{-A}^{+A} \mathrm{d}\tilde{x} K(x,y,\tilde{x},t). \quad (4.1)$$

Next, we note that since we will be interested in the on-shell value of the rotational component, we may set the observation point (x,y) as close to the flame front as we wish, i.e. $y \approx f(x,t)$, $[y > f(x,t)]$. The rotational component at such points is determined by a contribution coming from the integration over small τ and $\tilde{x} \approx x$. Indeed, taking the curl of (4.1), and using (3.2) yields

$$\varepsilon_{ik}\partial_i v_k = \int_{-A}^{+A} d\tilde{x} \int_0^{T+t} d\tau M(\tilde{x}, t-\tau)\delta(x - X[\tilde{x}, t, t-\tau])\delta(y - Y[\tilde{x}, t, t-\tau]),$$

which explicitly shows that the non-zero contribution to vorticity comes only from the point $\{\tilde{x}, \tau\}$ obeying the equations

$$x - X[\tilde{x}, t, t-\tau] = 0, \quad y - Y[\tilde{x}, t, t-\tau] = 0, \tag{4.2}$$

which for $y \approx f(x,t)$ means that the point (X, Y) is close to the flame front, and hence

$$\begin{aligned} X(\tilde{x}, t, t-\tau) &= \tilde{x} + w_+(\tilde{x}, t)\tau + O(\tau^2), \\ Y(\tilde{x}, t, t-\tau) &= f(\tilde{x}, t-\tau) + u_+(\tilde{x}, t)\tau + O(\tau^2). \end{aligned}$$

Expanding $f(\tilde{x}, t-\tau)$ to the first order in τ and omitting the symbols $O(\tau^2)$, we thus obtain the following approximate expression for the fluid particle trajectory

$$X(\tilde{x}, t, t-\tau) \approx \tilde{x} + w_+(\tilde{x}, t)\tau, \quad Y(\tilde{x}, t, t-\tau) \approx f(\tilde{x}, t) + \bar{u}_+(\tilde{x}, t)\tau. \tag{4.3}$$

It follows that if, instead of the exact ones, these expressions are used to calculate the kernel K, we still obtain the true distribution of vorticity near the flame front. Indeed, it has just been shown that any integration over values of $\{\tilde{x}, \tau\}$ not satisfying (4.2), where (4.3) is not valid either, gives rise to a purely potential contribution. In particular, property (b) of this contribution is preserved, as is property (a), since the right-hand side of (4.1) is divergence-free identically whatever the form of the kernel. Finally, it is not difficult to see that condition (c) is also satisfied. Indeed, the above transformation of exact trajectories into the straightened ones given by (4.3) leaves expression (4.1) bounded. Therefore, the potential field added in the course of this transformation is likewise bounded. It should be stressed that hereafter we will focus only on the on-shell value of the rotational component, so the meaning of the symbol $\overset{\circ}{=}$ must be further specified. Of course, this transformation of trajectories changes the bulk vorticity distribution, and thereby the velocity field downstream. However, both before and after transformation, integration over finite τs in (3.5) results in a field that is potential near the flame front. Both fields have a complicated structure which is unknown in general, but since they are potential near the front and bounded, we can use the on-shell value of their difference to define a field satisfying (a)–(c) in the whole downstream region. The existence of this field is guaranteed by the Cauchy theorem. To be precise, we use the Cauchy formula to construct the field satisfying (a)–(c) as the analytic function with the given boundary value. It is in this sense that the above transformation is said to respect properties (a)–(c). In particular, the symbol $\overset{\circ}{=}$ is used below to relate the on-shell values (or near-the-front values in the case of the integral kernel) of functions that differ by a field satisfying (a)–(c) downstream.

We now proceed to explicit evaluation of the integral kernel (3.5) which, after the transformations (4.3) are performed, has the form (when transforming the kernel $K(x, y, \tilde{x}, t)$, we use the same symbol $\overset{\circ}{=}$ to relate expressions that upon substitution in (4.1) give rise to fields that are equal in the sense of $\overset{\circ}{=}$)

$$K(x, y, \tilde{x}, t) \overset{\circ}{=} \frac{1}{\pi} \int_0^{T+t} d\tau M(\tilde{x}, t-\tau) \ln \left\{ \bar{v}_+^2 \tau^2 - 2(\boldsymbol{r} \cdot \bar{\boldsymbol{v}}_+)\tau + r^2 \right\}^{1/2}, \tag{4.4}$$

where $\boldsymbol{r} = (x - \tilde{x}, y - f(\tilde{x}, t))$. The integrand here involves $M(\tilde{x}, t - \tau)$ which is an unknown function of τ. In view of what has been said about the near-the-front structure of the rotational component, one might think that it would be sufficient to expand this function to the first order in τ, and then calculate the integral. However, this operation is not allowed as it would violate condition (c) and, as a result, would yield erroneous predictions (see § 5). In particular, the on-shell value of the rotational component cannot be found by setting $\tau = 0$ in the argument of M. There is no such problem in the case of steadily propagating flames, as M is then time-independent in a frame of reference attached to the front. To overcome this difficulty, we will explicitly extract the singular part of the τ-integral, which is related to the singularities of the logarithm located at the points

$$\tau_{\pm} = \frac{r}{\bar{v}_+}(\Omega \pm \mathrm{i}\sqrt{1 - \Omega^2}), \quad \Omega = \frac{(\boldsymbol{r} \cdot \bar{\boldsymbol{v}}_+)}{r\bar{v}_+} \tag{4.5}$$

in the complex τ-plane. For this purpose, we first formally integrate (4.4) by parts:

$$\begin{aligned} K(x, y, \tilde{x}, t) &\overset{\circ}{=} -\frac{1}{\pi}\int_0^{T+t} \mathrm{d}\left(\int_\tau^{T+t} \mathrm{d}\tau_1 M(\tilde{x}, t - \tau_1)\right) \ln\left\{\bar{v}_+^2\tau^2 - 2(\boldsymbol{r} \cdot \bar{\boldsymbol{v}}_+)\tau + r^2\right\}^{1/2} \\ &= \frac{\ln r}{\pi}\int_0^{T+t} \mathrm{d}\tau_1 M(\tilde{x}, t - \tau_1) \\ &\quad + \frac{1}{2\pi}\int_0^{T+t} \mathrm{d}\tau \left(\int_\tau^{T+t} \mathrm{d}\tau_1 M(\tilde{x}, t - \tau_1)\right)\left\{\frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-}\right\}. \end{aligned}$$

The first term on the right gives rise to a pure potential which is bounded, so it can be omitted. Denoting also

$$\int_\tau^{T+t} \mathrm{d}\tau_1 M(\tilde{x}, t - \tau_1) \equiv \mathscr{M}(\tilde{x}, \tau, t), \tag{4.6}$$

we thus have

$$K(x, y, \tilde{x}, t) \overset{\circ}{=} \frac{1}{2\pi}\int_0^{T+t} \mathrm{d}\tau \mathscr{M}(\tilde{x}, \tau, t)\left\{\frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-}\right\}.$$

To extract the singular part of this integral, we deform the contour of integration in the complex τ-plane so as to move it away from the poles; this necessarily implies certain timewise restrictions on the function $\mathscr{M}(\tilde{x}, \tau, t)$, and hence on $M(\tilde{x}, t)$. We shall return to this matter later (see § 7 and the Appendix). Here we only mention that in essence, the function $M(\tilde{x}, t)$ is required to be analytic in the vicinity of the real axis in the complex t-plane, which is guaranteed by the existence of a non-zero λ_c. To respect the reality of the kernel, we take the singular part as a half-sum of two expressions obtained respectively by deforming the contour above and below the real axis. The contribution we are interested in arises from moving the contours beyond the poles (4.5) (see figure 2).

According to the Cauchy theorem,

$$\begin{aligned} K(x, y, \tilde{x}, t) &\overset{\circ}{=} \frac{2\pi\mathrm{i}}{4\pi}\{\mathscr{M}(\tilde{x}, \tau_+, t) - \mathscr{M}(\tilde{x}, \tau_-, t)\} \\ &\quad + \frac{1}{4\pi}\int_{C_- \cup C_+} \mathrm{d}\tau \mathscr{M}(\tilde{x}, \tau, t)\left\{\frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-}\right\}. \end{aligned} \tag{4.7}$$

Instead of proving that the integral over the contour $C_- \cup C_+$ is free of singularity and gives rise to a bounded divergence-free potential, it is easier to show that the

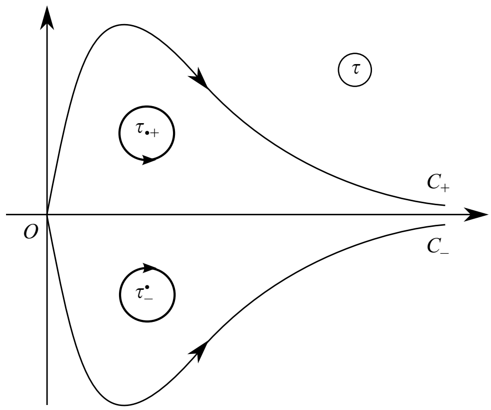

FIGURE 2. Extraction of the singularity in (4.4) by contour deformation in the complex τ-plane.

first term in (4.7) is bounded and correctly reproduces the vorticity distribution at the flame front (at this stage of transformation, property (a) is satisfied identically). This is done in the Appendix. Thus,

$$K(x, y, \tilde{x}, t) \overset{\circ}{=} \frac{\mathrm{i}}{2}\int_{\tau_+}^{\tau_-} \mathrm{d}\tau M(\tilde{x}, t-\tau). \tag{4.8}$$

Inserting (4.8) into (4.1) finally yields

$$v_i^v \overset{\circ}{=} \frac{\mathrm{i}}{4}\varepsilon_{ik}\partial_k \int_{-A}^{+A} \mathrm{d}\tilde{x} \int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau). \tag{4.9}$$

Since $\tau_+^* = \tau_-$, the right-hand side of this relation is real.

To make the meaning of the above calculation qualitatively more vivid, it is useful to mention an interrelation between the roles played by conditions (a)–(c) in the above derivation. When calculating the near-the-front value of kernel $K(x, y, \tilde{x}, t)$, we retain terms of first order with respect to τ. This is sufficient for the calculation of the rotational component of velocity at the front, taking into account that this quantity is determined by the first spatial derivatives of the kernel, and that $r = \tau v_+$ at the point defined by (4.2). This means, in particular, that τ in the function $M(\tilde{x}, t-\tau)$ may not be neglected. Moreover, as was mentioned above, this function cannot be expanded in τ without violating (c): the condition $M(\tilde{x}, t) \to 0$ for $t \to -\infty$ guarantees convergence of the τ-integral in the limit $T \to \infty$. At the same time, it is seen from (4.9) that as a result of the τ-integration, the dependence of the function $M(\tilde{x}, t-\tau)$ on τ is transmuted into a dependence on the coordinates. This dependence does not affect the vorticity distribution along the front, because $\tau = r/v_+$ along the streamlines, so that $\tau = 0$ when the observation point is placed on the front ($r = 0$). Thus, the seemingly innocent condition (c) entails a non-trivial change in the structure of the potential component of the burnt-gas velocity in comparison with that of the steady regime. In the latter case, the condition $M(\tilde{x}, t) \to 0$ for $t \to -\infty$ does not apply, but since the M-function is independent of time, it remains independent of the coordinates (x, y) at all stages of the calculation. Proceeding then as in Kazakov (2005*b*), one can verify that the divergent contribution to the velocity field, arising from integration over large τ, is also coordinate-independent, so that condition (c) is still met. Finally, it is not difficult to show that expression (4.9) cannot be further simplified, following the lines of Kazakov (2005*b*), by omitting the additional potential contribution *after* the spatial differentiation: it turns out that this contribution satisfies condition (a) only in the steady case. We shall return to this point in § 6.1.

4.1. *Definition of the rotational component*

Having obtained an explicit expression for the rotational component of the burnt-gas velocity for a finite Σ, we now need to consider the question of the transition to the limits $A \to \infty$, $T \to \infty$. Generally, the rule to be taken for these limits depends on the problem under consideration. In the case of unsteadily propagating flames, this issue is complicated by the fact that the expression found for the rotational component is essentially non-local, both in space and time. The time non-locality explicitly shows itself through τ-integration in (4.9), and is to be expected from the very outset. In fact, appearance of time non-locality is *inevitable*, because there must exist some mechanism transferring the influence of flame history onto its current state. Such a mechanism is unnecessary only in the stationary case, where all information about the flame history is, in a sense, left at infinity downstream. Furthermore, we have seen that the time dependence of the function $M(\tilde{x}, t)$ is partially transmuted into coordinate dependence, so the time non-locality naturally entails an essential spatial non-locality. This is again in contrast with the steady case, where it turned out to be possible to find a local on-shell expression for the rotational component (cf. equation (37) in Kazakov 2005*b*).

Although the parameter T does not explicitly appear in (4.9), it should be borne in mind that in the course of derivation of this expression, the time dependence of the function $M(\tilde{x}, t)$ function has been transmuted into a dependence on the spatial coordinates. Hence, in order to preserve property (c) of the potential component, the limit $T \to \infty$ is generally to be taken under the assumption that the function $M(\tilde{x}, t)$ must vanish for $t \to -\infty$. Concerning the limit $A \to \infty$, such an assumption would be irrelevant because of flame periodicity along the x-axis. To ensure convergence we introduce an intermediate regularization of the x-integral, replacing (4.9) by

$$v_i^v(\mu) \overset{\circ}{=} \frac{\mathrm{i}}{4}\varepsilon_{ik}\int_{-A}^{+A} \mathrm{d}\tilde{x}\mathrm{e}^{-\mu r}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau), \tag{4.10}$$

where $\mu > 0$ is a sufficiently large parameter. We then take the limit $A \to \infty$ and *define* the rotational component as *the analytic* continuation of (4.10) to the value $\mu = 0$ along the real axis in the complex μ-plane. Replacing also the symbol $\overset{\circ}{=}$ by an equality, the definition thus becomes

$$v_i^v = \frac{\mathrm{i}}{4}\varepsilon_{ik}\left\{\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\mathrm{e}^{-\mu r}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau)\right\}_{\mu=0^+}. \tag{4.11}$$

We will now show that this definition respects properties (a)–(c). Rewriting (4.11) as

$$v_i^v = \frac{\mathrm{i}}{4}\varepsilon_{ik}\int_{-A_0}^{+A_0} \mathrm{d}\tilde{x}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau)$$
$$+ \frac{\mathrm{i}}{4}\varepsilon_{ik}\left\{\left[\int_{-\infty}^{-A_0} + \int_{+A_0}^{+\infty}\right] \mathrm{d}\tilde{x}\mathrm{e}^{-\mu r}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau)\right\}_{\mu=0^+},$$

where $A_0 > 0$ is arbitrary, and comparing with (4.9) one sees that taking the limit $A \to \infty$ followed by the analytic continuation in μ does not change the vorticity distribution in the arbitrarily large domain $|x| < A_0$. Therefore, the above analytical operations are equivalent to the addition of some potential field, so that (b) is met.

Furthermore, this field is bounded in the sense of (c). To see this, consider the quantity

$$\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t-\tau). \tag{4.12}$$

Using (4.5) and the Newton–Leibnitz formula, it can be seen that this expression is a combination of the functions $M(\tilde{x}, t-\tau_\pm)$ times spatial derivatives of $\tau_\pm$. It ensues from (4.5) that $|\tau_\pm| = r/\bar{v}_+ \to \infty$ for $|\tilde{x}| \to \infty$. Hence, if we assume that $M(\tilde{x}, t)$ is exponentially bounded in the vicinity of the point $t = \infty$ in the complex t-plane, i.e. $|M(\tilde{x}, t)| < e^{c|t|}$ for some $c > 0$ and $|t| \to \infty$, there then exists a large enough value of μ so that the $\tilde{x}$-integral in (4.11) converges. On the other hand, the function $M(\tilde{x}, t)$ is periodic with respect to $\tilde{x}$ as a result of flow periodicity. Therefore, all the singularities of the expression in the curly brackets in (4.11) are off the real axis in the complex μ-plane, except possibly for a simple pole at $\mu = 0$. This latter corresponds to an additive constant, B_k, in quantity (4.12). The appearance of such a term is not forbidden by the requirement of periodicity in $\tilde{x}$. B_k is independent of $\boldsymbol{x}$ by virtue of the same flow periodicity. But after $\tilde{x}$-integration this term gives rise to a contribution of the form $B_k/\mu + h_k(x)$, where $h_k(x)$ vanishes for $\mu \to 0$. Since B_k/μ disappears upon spatial differentiation, Dv_i^v can be continued analytically to $\mu = 0$, so property (c) is indeed fulfilled. Finally, the divergence of the rotational component

$$\operatorname{div} \boldsymbol{v}^v = -\frac{i}{4}\left\{\mu \int_{-\infty}^{+\infty} d\tilde{x} e^{-\mu r}\frac{r_i}{r}\varepsilon_{ik}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} d\tau M(\tilde{x}, t-\tau)\right\}_{\mu=0^+}$$

generally does not vanish. We note, however, that $r_i/r = -\text{sign}(\tilde{x})\delta_{1i} + O(1/|\tilde{x}|)$ for $|\tilde{x}| \to \infty$. Therefore, the only term in the $\tilde{x}$-integral that remains after continuation to $\mu = 0$, is an $\boldsymbol{x}$-independent constant proportional to $(\delta_{1i}\varepsilon_{ik}B_k/\mu)$. Thus, $\operatorname{div} \boldsymbol{v}^v = \text{const.}$, and condition (a) is satisfied.

To conclude this section, let us make a comment on the use of intermediate regularization in the definition of rotational component. The exponential damping used above is the simplest and natural choice which is also most convenient in actual calculations to be performed in §6. Of course, this choice is not unique, and different regularization schemes will generally lead to different expressions for the rotational component. However, if the given scheme respects conditions (a)–(c), then the corresponding expression for $\boldsymbol{v}^v$ will be equivalent to (4.11) in the sense of $\overset{\circ}{=}$. Thus, the choice of the regularization is a matter of convenience, and $\exp(-\mu r)$ in the above equations can be replaced by any integrable function $E(\mu r)$ satisfying $E(0) = 1$ and $E(+\infty) = 0$, provided it preserves properties (a)–(c).

5. Closed description of non-stationary flames

We are now in a position to write down an integro-differential equation relating the on-shell values of fresh-fuel velocity and the front position function. Let us introduce the complex variable $z = x + iy$, and the complex velocity $\omega = u + iw$. By virtue of properties (a), (b) the complex quantity $d\omega^p/dz$, where $\omega^p = u^p + iw^p$, is an analytical function of the complex variable z in the downstream region. In conjunction with property (c), analyticity of $d\omega^p/dz$ can be expressed in the form of the following dispersion relation (Kazakov 2005*a*, *b*):

$$(1 + i\hat{\mathscr{H}})(\omega_+^p)' = 0, \tag{5.1}$$

where the prime denotes x-differentiation, and the action of the operator $\hat{\mathscr{H}}$ on an arbitrary function $a(x)$ is defined by

$$(\hat{\mathscr{H}}a)(x) = \frac{1 + \mathrm{i}f'(x,t)}{\pi} \fint_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\, \frac{a(\tilde{x})}{\tilde{x} - x + \mathrm{i}[f(\tilde{x},t) - f(x,t)]}, \tag{5.2}$$

where the slash denotes the principal value of the integral. $\hat{\mathscr{H}}$ has properties similar to the Hilbert operator $\hat{H}$ (and is effectively the Hilbert transform along the front). In particular, it was proved by Kazakov (2005*b*) that

$$\hat{\mathscr{H}} \circ \hat{\mathscr{H}} = -1. \tag{5.3}$$

The identity (5.1) relates, in a complicated way, the on-shell values of the burnt-gas velocity and the flame-front position. The fresh-gas velocity also satisfies conditions (a)–(c), this time in the upstream region. Indeed, (a) in this case is just the derivative of (2.2), (b) follows from the Thomson theorem and the boundary conditions upstream, and (c) is true because fresh-gas velocity is bounded. The consequence of these properties is the following dispersion relation for $\omega_- = u_- + \mathrm{i}w_-$

$$(1 - \mathrm{i}\hat{\mathscr{H}})\,(\omega_-)' = 0. \tag{5.4}$$

Let $[\boldsymbol{v}]$ denote the jump of the gas velocity across the flame front, $[\boldsymbol{v}] = \boldsymbol{v}(x, f(x,t) + 0) - \boldsymbol{v}(x, f(x,t) - 0)$. Then the sought-after equation for ω_-, f is obtained by substituting

$$\omega_+^p = -\omega_+^v + \omega_- + [\omega]$$

in (5.1), and using (4.11), (5.4)

$$2(\omega_-)' + (1 + \mathrm{i}\hat{\mathscr{H}}) \left\{ [\omega] - \frac{\mathrm{i}}{4} \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x} e_k \frac{\partial}{\partial x_k} \int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t - \tau) \right\}' = 0, \tag{5.5}$$

where $e_k = \varepsilon_{2k} + \mathrm{i}\varepsilon_{1k}$, and for brevity we omit the regularizing factor $\mathrm{e}^{-\mu r}$ in the integrand as well as the accompanying symbol of analytic continuation. In the last term on the left, the argument y is understood to be set equal to $f(x,t)$ after partial spatial differentiation is performed, but before the x-differentiation denoted by the prime. The value of vorticity at the front and the normal velocity of the burnt gas, entering the function $M(\tilde{x}, t - \tau)$, as well as the velocity jumps at the front are all known functionals of on-shell fresh-gas velocity (Matalon & Matkowsky 1982; Pelce & Clavin 1982), taking into account the effects of finite flame thickness. For instance, the dependence of the pressure jump on the Lewis number (heat-to-mass diffusivity ratio) shows itself through σ_+; gravity effects appear in the same way. For zero-thickness flame fronts one has

$$\bar{v}_+^n = \theta, \quad [u] = \frac{\theta - 1}{N}, \quad [w] = -f' \frac{\theta - 1}{N}, \tag{5.6}$$

$$\sigma_+ = -\frac{\theta - 1}{\theta N} \left\{ \frac{\mathrm{D}w_-}{\mathrm{D}t} + f' \frac{\mathrm{D}u_-}{\mathrm{D}t} + \frac{1}{N} \frac{\mathrm{D}f'}{\mathrm{D}t} \right\}, \tag{5.7}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \left(w_- + \frac{f'}{N} \right) \frac{\partial}{\partial x}.$$

Thus, the complex equation (5.5) gives two equations for three functions $w_-(x,t)$, $u_-(x,t)$ and $f(x,t)$. Together with the evolution equation

$$(\bar{\boldsymbol{v}}_- \cdot \boldsymbol{n}) = 1 + S(u_-, w_-, f'), \tag{5.8}$$

where S is a known functional of its arguments, proportional to the flame-front thickness (or, rather, the cutoff wavelength λ_c), (5.5) provides a closed description of unsteady flames in the most general form. Its application to various particular problems is given in the next section.

Before proceeding, the following point is worth commenting on. In the analysis that led us to (5.5), the front was represented by a single-valued function $y = f(x, t)$, which excludes the overhangs or fronts that double back on themselves. This over-restrictive assumption, adopted so far for simplicity, can be relaxed as follows. Let us parameterize the flame front by a real parameter ξ so that $(x(\xi, t), y(\xi, t))$ represents a diffeomorphic mapping of the interval $-\infty < \xi < +\infty$ onto the front at time instant t. Then, setting $x(\xi, t) + \mathrm{i}y(\xi, t) = Z(\xi, t)$, we define a new metric coefficient $N(\xi, t)$ in terms of the infinitesimal arclength along the front, $\mathrm{d}l$, by $\mathrm{d}l = N(\xi, t)\,\mathrm{d}\xi$; in other words, $N(\xi, t) = |\partial_\xi Z(\xi, t)|$. As long as the fresh-gas and burnt-gas regions remain connected domains, the Cauchy theorem guarantees the existence of a generalized operator $\hat{\mathscr{H}}$ such that $(1 \pm \mathrm{i}\hat{\mathscr{H}})\mathrm{d}\omega_\pm/\mathrm{d}\xi = 0$. Specifically, when acting on a smooth function $a(\xi, t)$, the new $\hat{\mathscr{H}}$ produces

$$(\hat{\mathscr{H}}a)(\xi, t) = \frac{\partial_\xi Z}{\pi} \fint \frac{a(\tilde{\xi}, t)\,\mathrm{d}\tilde{\xi}}{Z(\tilde{\xi}, t) - Z(\xi, t)}, \tag{5.9}$$

instead of (5.2). Accordingly, if $M(\xi, t)$ is still defined as $M = N(\xi, t)\,\sigma_+(\xi, t)\,\bar{u}^n_+(\xi, t)$, (5.5) is formally unchanged, except for the fact that the prime now denotes $\mathrm{d}/\mathrm{d}\xi$, and the time derivatives must be handled in a way consistent with the new representation of the flame front. However, such a reparameterization still does not capture what happens when isolated pockets of fresh gas form, because the fresh domain then ceases to be path-connected. Unfortunately, as long as a *local* propagation law (cf. (5.8)) is employed, such a phenomenon cannot be excluded *a priori*: there is no way that a flame element can ‘know’ that another one is going to produce a ‘head-on’ collision.

The above reparameterization is unnecessary for the case of wrinkled fronts considered below.

6. Equations (5.5) and (5.8) in limiting cases

To provide a consistency check for (5.5) and also to gain a deeper insight into the structure of this equation, we use it below to derive anew the classical results on flame-front dynamics.

6.1. *Darrieus–Landau instability of zero-thickness flames*

Let us first consider the classical linear stability problem of zero-thickness planar flame propagation (Darrieus 1938; Landau 1944). In this case, $\bar{v}^n_+ = \theta$, $N = 1$, while the linearized on-shell vorticity (5.7) reads

$$\sigma_+ = -\frac{\theta - 1}{\theta}\left(\frac{\partial w_-}{\partial t} + \frac{\partial^2 f}{\partial t \partial x}\right).$$

Accordingly, expression (4.11) simplifies to

$$v^v_i = -\frac{\mathrm{i}(\theta - 1)}{4}\varepsilon_{ik}\left\{\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\,\mathrm{e}^{-\mu|x-\tilde{x}|}\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau\left(\frac{\partial w_-}{\partial t} + \frac{\partial^2 f}{\partial t \partial x}\right)\right\}_{\mu = 0^+}. \tag{6.1}$$

Since the integrand here is a first-order quantity, it is sufficient to calculate $\tau_{\pm}$ for a plane front that is assumed to be at $y=0$

$$\tau_{\pm}=\frac{y}{\theta}\pm\mathrm{i}\frac{|x-\tilde{x}|}{\theta}. \tag{6.2}$$

Let the disturbance be periodic in x, growing exponentially with time. Spatial periodicity of the linear problem is most conveniently represented in the complex form, in which case

$$f(x,t), u_{-}(x,t), w_{-}(x,t)\sim \mathrm{e}^{\mathrm{i}kx+\nu t}. \tag{6.3}$$

However, it should be borne in mind that the coefficients in (5.5) are also complex. In order to preserve the correct complex structure of this equation, all the functions involved need to be written in a form containing no imaginary coefficients. To find v_i^v, we must evaluate the following integrals

$$I_k(x,y,t,\mu)=\int_{-\infty}^{+\infty}\mathrm{d}\tilde{x}\mathrm{e}^{-\mu|x-\tilde{x}|}\frac{\partial}{\partial x_k}\int_{y/\theta-\mathrm{i}|x-\tilde{x}|/\theta}^{y/\theta+\mathrm{i}|x-\tilde{x}|/\theta}\mathrm{d}\tau\mathrm{e}^{\mathrm{i}k\tilde{x}+\nu(t-\tau)}$$

for $\mu>0$ and $k=1,2$. A straightforward calculation gives

$$\begin{aligned} I_1(x,y,t,\mu) &= \frac{\mathrm{i}}{\theta}\mathrm{e}^{\mathrm{i}kx+\nu t-\nu y/\theta}\left\{\left(\frac{1}{\mu+\mathrm{i}k+\mathrm{i}\nu/\theta}+\frac{1}{-\mu+\mathrm{i}k+\mathrm{i}\nu/\theta}\right)+(\nu\rightarrow-\nu)\right\}, \\ I_2(x,y,t,\mu) &= \frac{1}{\theta}\mathrm{e}^{\mathrm{i}kx+\nu t-\nu y/\theta}\left\{\left(\frac{1}{\mu+\mathrm{i}k+\mathrm{i}\nu/\theta}+\frac{1}{-\mu+\mathrm{i}k+\mathrm{i}\nu/\theta}\right)-(\nu\rightarrow-\nu)\right\}, \end{aligned}$$

where $(\nu\rightarrow-\nu)$ is shorthand for the preceding parenthesis with ν changed into its opposite. The on-shell values of these functions analytically continued to $\mu=0$ are

$$I_1(x,0,t,0)=\frac{4}{\theta}\mathrm{e}^{\mathrm{i}kx+\nu t}\frac{k}{k^2-\nu^2/\theta^2},\quad I_2(x,0,t,0)=\frac{4\mathrm{i}}{\theta}\mathrm{e}^{\mathrm{i}kx+\nu t}\frac{\nu/\theta}{k^2-\nu^2/\theta^2}. \tag{6.4}$$

Using this in (6.1) yields

$$w_{+}^{v} = \frac{\nu/\theta}{k^2-\nu^2/\theta^2}\frac{(\theta-1)}{\theta}(\nu w_{-}+\mathrm{i}k\nu f), \tag{6.5}$$

$$u_{+}^{v} = \frac{\mathrm{i}k}{k^2-\nu^2/\theta^2}\frac{(\theta-1)}{\theta}(\nu w_{-}+\mathrm{i}k\nu f). \tag{6.6}$$

To put these expressions in an explicitly real form, it suffices to write

$$w_{+}^{v}=-\frac{\sigma_{+}\nu/\theta}{k^2-\nu^2/\theta^2},\quad u_{+}^{v}=-\frac{\sigma_{+}'}{k^2-\nu^2/\theta^2}.$$

Next, we note that in the linear approximation the operator $\hat{\mathscr{H}}$ becomes just the Hilbert operator $\hat{H}$, whose action on the harmonic functions is defined by

$$\hat{H}\exp(\mathrm{i}kx)=\mathrm{i}\chi(k)\exp(\mathrm{i}kx), \tag{6.7}$$

where

$$\chi(k)=\begin{cases}+1, & k>0,\\ 0, & k=0,\\ -1, & k<0.\end{cases}$$

Equation (5.5) thus becomes

$$2(\omega_-)' + (1 + \mathrm{i}\hat{H})\left\{[\omega] + \frac{\sigma_+' + \mathrm{i}\sigma_+\nu/\theta}{k^2 - \nu^2/\theta^2}\right\}' = 0. \tag{6.8}$$

Extracting the real and imaginary parts of this equation, we find

$$2(u_-)' + \left\{[u] - \hat{H}[w] + \frac{\sigma_+' - \hat{H}\sigma_+\nu/\theta}{k^2 - \nu^2/\theta^2}\right\}' = 0, \tag{6.9}$$

$$(w_-)' = \hat{H}(u_-)'. \tag{6.10}$$

Finally, upon linearization the jump conditions (5.6) simplify to

$$[u] = \theta - 1,\ [w] = -f'(\theta - 1), \tag{6.11}$$

while the linearized evolution equation reads

$$u_- - \frac{\partial f}{\partial t} = 1. \tag{6.12}$$

Inserting these into (6.9) and then substituting $\sigma_+ = -(\theta - 1)/\theta(\mathrm{i}\chi(k)\nu^2 f + \mathrm{i}k\nu f)$ leads after some simple algebra to the equation

$$\frac{\theta + 1}{\theta}\nu^2 + 2\nu|k| - (\theta - 1)k^2 = 0, \tag{6.13}$$

which is nothing but the well-known Darrieus–Landau dispersion relation expressing the perturbation growth rate as a function of the wavenumber and the gas expansion coefficient (Darrieus 1938; Landau 1944)

$$\nu = \frac{\theta}{\theta + 1}\left(\sqrt{1 + \theta - \frac{1}{\theta}} - 1\right)|k|. \tag{6.14}$$

The effects related to finite front thickness can be explicitly accounted for in the linear analysis above by including terms linear in f'' on the right-hand side of (6.11), (6.12). This would change the last term in (6.13) by the extra factor $(1 - |k|\,\lambda_c/2\pi)$, and make ν roughly parabolic for all $|k| \leqslant 2\pi/\lambda_c$; $\max(\nu)$ gives an estimation of the typical growth time as $t_{DL} \simeq 2\lambda_c/\pi(\sqrt{\theta} - 1)$, to be used later (§ 6.4).

Next, let us return to the remark made after (4.4). It was mentioned there that even if one is interested only in the on-shell values of the rotational component, the τ-dependence of the function $M(\tilde{x}, t - \tau)$ cannot be neglected. Evidently, doing so is only a matter of rewriting the equation obtained by Kazakov (2005*a*, *b*) for steady flames in terms of the local current flow velocity relative to the front. It is not difficult to verify that in the case under consideration, this would change the coefficient of ν^2 in (6.13) to the wrong value $(\theta - 1)/\theta$. This change is thus a reflection of the memory effects encoded in the function $M(\tilde{x}, t - \tau)$. Equation (5.5) properly takes into account these effects, correctly reproducing the Darrieus–Landau relation, and automatically captures all the aspects of flame dynamics pertaining to inertia.

At this stage of the analysis it is appropriate to discuss the meaning of the analytic continuation appearing in the definition (4.11) in more detail. This continuation is used to render the $\tilde{x}$-integral meaningful in the limit $A \to \infty$. One can avoid using this analytic procedure if the formal result of improper integration along the infinite flame front is treated in the sense of distributions. Indeed, in this case expressions (6.4) for the rotational component would contain additional terms proportional to $\delta(k + \nu/\theta)$

or $\delta(k - \nu/\theta)$ coming from the integration over large $\tilde{x}s$. To see that these terms are inconsequential, we wish to recall that the above consideration of the evolution of a single k-mode is not completely adequate from a physical point of view, because in practice one always deals not with an infinite single mode, but rather with finite wave packets containing a continuum of wavenumbers k. This means that the physical expression for the flame-front position with the given k_0 is obtained by integrating the found solution $f(x, t)$ with some weight over a small but finite range Δk around k_0. Upon this integration all terms involving $\delta(k \pm \nu/\theta)$ disappear, because $\nu = \pm k\theta$ are not roots of the Darrieus–Landau relation. The two procedures are consequently equivalent.

6.2. *The small* $(\theta - 1)$ *expansion*

Let us next consider the case of small gas expansion. We will verify that within the framework of the asymptotic expansion with respect to $\theta - 1 \equiv \alpha$, (5.5) reduces at the first post-Sivashinsky order to the well-known Sivashinsky–Clavin equation (Sivashinsky & Clavin 1987)†. To perform the asymptotic expansion we recall that the cutoff wavelength for the short wavelength perturbations λ_c is of the order $1/\alpha$. This means that the wavenumbers involved are $O(\alpha)$. In other words, spatial differentiation of a flow variable raises its order by 1; in particular, $f' = O(\alpha)$. Also, (6.14) tells us that for small α, the perturbation growth rate $\nu = k\alpha/2 = O(\alpha^2)$, so that the order of a flow variable is raised by 2 upon time differentiation. It then follows from (5.8) (with $S \equiv 0$) that $u_- = 1 + O(\alpha^2)$, while potentiality of the upstream flow implies that $w_- = O(\alpha^2)$ (this is clearly seen from the dispersion relation (5.4)). The first post-Sivashinsky approximation corresponds to retaining terms of fourth order in (5.5), or equivalently, $O(\alpha^3)$-terms before the spatial differentiation. It was shown by Kazakov (2005*b*) that to this order, $\hat{\mathscr{H}}$ becomes just the Hilbert operator. Therefore, as in the linear case considered above, the real and imaginary parts of (5.5) are readily separated to give

$$2\,(u_-)' + \left\{ [u] - \hat{H}[w] + (\varepsilon_{1k}\hat{H} - \varepsilon_{2k})\frac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\,\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau\, M(\tilde{x}, t - \tau) \right\}' = 0, \quad (6.15)$$

$$(w_-)' = \hat{H}(u_-)'. \quad (6.16)$$

Since the quantity $J_k \equiv \partial_k \int_{\tau_-}^{\tau_+} \mathrm{d}\tau M$ is under the sign of the spatial integral, we need the function $M(\tilde{x}, t)$ to fourth order in α. To this accuracy, it is equal to the on-shell vorticity

$$M = \sigma_+ = -(\theta - 1)(\dot{f}' + f'f''),$$

† It is worth noting that the small expansion parameter used by Sivashinsky and Clavin is $\gamma \equiv (\theta - 1)/\theta = \alpha/(1 + \alpha)$, rather than α. The reason for switching from α to γ, sometimes found in the literature, is that the latter would improve expansion accuracy/convergence. The argument given in this connection, namely, that $\gamma < 1$ for all θ, while α is small only for θ close to 1, is not quite correct. The validity of an expansion is determined not by the value of the expansion parameter, but rather by the relative value of the terms neglected in the course of the calculation. It is true that expanding the decreasing function $f(\theta) = 1/\theta = 1/(1+\alpha)$ in terms of $\alpha/(1+\alpha)$ instead of α is an improvement. However, it is not true in the case of the increasing function $f(\theta) = \theta - 1 = \alpha$ which certainly appears in the governing equations (see (5.5), (5.6)). Thus, whether or not a change of the expansion parameter improves the expansion accuracy/convergence is a question involving the structure of the whole perturbation series, which cannot be resolved from knowledge of its first few terms. One of the goals of our approach is to make questions of this kind accessible for theoretical analysis.

the dot denoting differentiation with respect to time t. Since this quantity is already $O(\alpha^4)$, $\tau_\pm$ can be taken in the form (6.2) with $\theta = 1$. Thus, performing spatial differentiation and setting $y = 0$ gives

$$\begin{aligned} J_1 &= \mathrm{i}\chi(x-\tilde{x})\left\{\sigma_+(\tilde{x}, t-\mathrm{i}|x-\tilde{x}|) + \sigma_+(\tilde{x}, t+\mathrm{i}|x-\tilde{x}|)\right\}, \\ J_2 &= \left\{\sigma_+(\tilde{x}, t-\mathrm{i}|x-\tilde{x}|) - \sigma_+(\tilde{x}, t+\mathrm{i}|x-\tilde{x}|)\right\}. \end{aligned}$$

We now show that the imaginary parts in the argument of σ_+ can be omitted. Note that within the asymptotic expansion in α, the dependence of σ_+ on $\mathrm{i}|x-\tilde{x}|$ can be treated perturbatively. Indeed, let us write formally

$$\sigma_+(\tilde{x}, t-\mathrm{i}|x-\tilde{x}|) = \sigma_+(\tilde{x}, t) - \mathrm{i}|x-\tilde{x}|\dot{\sigma}_+(\tilde{x}, t) - \tfrac{1}{2}(x-\tilde{x})^2\ddot{\sigma}_+(\tilde{x}, t) + \cdots.$$

To assess the relative order of consecutive terms in this series, we have first to eliminate the factors containing explicit coordinate dependence, which can be done through successive integration by parts with respect to $\tilde{x}$. One has, for example,

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}|x-\tilde{x}|\dot{\sigma}_+(\tilde{x}, t) &= \int_{-\infty}^{+\infty} \mathrm{d}\left(\int^{\tilde{x}} \mathrm{d}x_1\dot{\sigma}_+(x_1, t)\right)|x-\tilde{x}|, \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\chi(x-\tilde{x})\left(\int^{\tilde{x}} \mathrm{d}x_1\dot{\sigma}_+(x_1, t)\right), \end{aligned}$$

where, according to the discussion in the preceding section, the contributions of infinitely remote parts of the front have been omitted: as before, this prescription is easily seen to be equivalent to the analytic continuation using μ-regularization. Since each time differentiation adds two powers of α, while spatial integration subtracts only one, we may conclude that the above expansion for σ_+ is effectively an asymptotic series in powers of α. Hence, to the fourth order in α

$$J_1 = 2\mathrm{i}\chi(x-\tilde{x})\sigma_+(\tilde{x}, t), \quad J_2 = 0,$$

and so

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x} J_1 &= -2\mathrm{i}(\theta-1)\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\chi(x-\tilde{x})\left(\dot{f}' + f'f''\right)(\tilde{x}, t) \\ &= -4\mathrm{i}(\theta-1)\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\delta(x-\tilde{x})\left(\dot{f} + \frac{f'^2}{2}\right)(\tilde{x}, t) \\ &= -4\mathrm{i}(\theta-1)\left(\dot{f} + \frac{f'^2}{2}\right)(x, t), \end{aligned}$$

where the boundary terms at $\tilde{x} = \pm\infty$ were again omitted during the integration by parts, and the relation $\partial\chi(x-\tilde{x})/\partial\tilde{x} = -2\delta(x-\tilde{x})$ was used. Inserting this together with the velocity jumps into (6.15), (6.16) yields within the given accuracy

$$2u_- + (\theta-1)(\hat{H}f' + \dot{f}) = C_1(t), \tag{6.17}$$

$$2w_- + (\theta-1)(-f' + \hat{H}\dot{f}) = C_2(t), \tag{6.18}$$

where $C_{1,2}(t)$ are two 'integration constants' (some x-independent functions of time). Since the left-hand side of (6.18) is odd in x, one has $C_2 \equiv 0$. Using these formulas in the evolution equation (5.8) written in the form

$$u_- - \dot{f} - f'w_- = 1 + \frac{f'^2}{2}$$

leads to an equation for the function $f(x,t)$

$$\frac{\theta+1}{2}\dot{f}+\frac{\theta}{2}f'^2+C(t)=-\frac{\theta-1}{2}\hat{H}f', \tag{6.19}$$

where $C(t)=1-C_1(t)/2$. The function $C(t)$ can be found by averaging the obtained equation along the flame front:

$$C(t)=-\frac{\theta+1}{2}\langle\dot{f}\rangle-\frac{\theta}{2}\langle f'^2\rangle.$$

Equation (6.19) is nothing but the Sivashinsky–Clavin equation (Sivashinsky & Clavin 1987), with the term $C(t)$ added according to Joulin & Cambray (1992). Taking into account the transport processes inside the front would have added a term proportional to $\lambda_c f''$ to the right-hand side of (6.19).

Finally, it is interesting to note that memory effects are insignificant not only at the lowest order of the small-$(\theta-1)$ expansion, but also at the first post-Sivashinsky approximation considered here. In fact, a direct calculation shows that replacing $M(\tilde{x},t-\tau)$ by $M(\tilde{x},t)$ in (5.5) does not change (6.19). This is natural, because the condition $\alpha\ll 1$ implies slow dynamics. However, this is already not the case at the next order, which is possibly one reason why memory effects are often overlooked.

6.3. *Flame propagation in a time-dependent gravitational field*

Depending on the direction of flame propagation, a gravitational field leads to either strengthening or damping of the Darrieus–Landau instability. Moreover, the influence of sound waves on the front dynamics can be effectively described as flame propagation in a time-dependent gravitational field (Markstein 1964; Searby & Rochwerger 1991). Let $g(t)$ denote its strength, with the convention that $g>0$ corresponds to a stabilizing effect (see (6.23)). Inclusion of a gravitational field does not change the flow equations (2.2), (2.3), so that their consequence, (5.5), retains the same structure. The jump conditions (5.6) for the gas velocity are likewise left intact by gravity. The only place where $g(t)$ appears in our approach is in the expression of the on-shell vorticity (as a result of baroclinic effects inside the front). Namely, the gravity-induced jump in this quantity to be added to the right-hand side of (5.7) is (Hayes 1957)

$$\Delta\sigma_+=-\frac{(\theta-1)}{\theta N}g(t)f'(x,t). \tag{6.20}$$

If the development of the Darrieus–Landau instability is suppressed by the gravitational field, then it is natural to consider harmonic front perturbations rather than the exponentially growing ones used in §6.1. The choice between the two representations is just a matter of convenience. It will be seen below that although the intermediate procedure of analytic continuation in μ is somewhat different for imaginary ν, the final equations for the front position are just analytic continuations of one another with respect to frequency.

Because of the time dependence of $g(t)$, the function $f(x,t)$ is now to be taken as a superposition of an arbitrary number of harmonics

$$f(x,t)=\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{d}k f(k,\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}. \tag{6.21}$$

It is also convenient to introduce the Fourier decomposition of the function $G(x,t)=g(t)f'(x,t)$

$$G(x,t)=\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\mathrm{d}k G(k,\omega)\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}. \tag{6.22}$$

To perform an analytic continuation with respect to μ in (4.11), we will assume that $f(k,\omega)$ and $G(k,\omega)$, considered as functions of ω, vanish outside an arbitrarily large but finite frequency band $|\omega| \leqslant \omega_0$. Then, choosing $\mu > \omega_0/\theta$, inserting the above Fourier decompositions in the linearized expression for the on-shell vorticity, performing $\tilde{x}$-integration in (4.11) as before, and continuing the result analytically to $\mu = 0$, we find

$$w_+^v = -\mathrm{i}\frac{\theta-1}{\theta}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\frac{\omega/\theta}{k^2+\omega^2/\theta^2}\left\{(-\mathrm{i}\chi(k)\omega^2+k\omega)f(k,\omega)+G(k,\omega)\right\},$$

$$u_+^v = \mathrm{i}\frac{\theta-1}{\theta}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\frac{k}{k^2+\omega^2/\theta^2}\left\{(-\mathrm{i}\chi(k)\omega^2+k\omega)f(k,\omega)+G(k,\omega)\right\}.$$

The first terms on the right-hand sides of these equations are just analytic continuations of expressions (6.5), (6.6) to the imaginary value of the growth rate: $\nu \to -i\omega$. As in the Darrieus–Landau problem, (5.5) splits into two real equations

$$2(u_-)' + \{[u] - \hat{H}[w] - u_+^v + \hat{H}w_+^v\}' = 0, \quad (w_-)' = \hat{H}(u_-)',$$

which after substitution of the expressions obtained for v_i^k together with (6.11), (6.12) yield

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}\left\{\left[\frac{\theta+1}{\theta}\omega^2 + 2\mathrm{i}\omega|k| + (\theta-1)k^2\right]kf(k,\omega) + \frac{\theta-1}{\theta}\mathrm{i}k\chi(k)G(k,\omega)\right\}\left(\frac{\omega}{\theta}+\mathrm{i}k\right)^{-1} = 0.$$

Acting on this equation by the operator

$$\left(\frac{\mathrm{i}}{\theta}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\int \mathrm{d}x,$$

and taking into account definitions (6.21), (6.22), we arrive at the well-known equation for the front position (Markstein 1964)

$$\frac{\theta+1}{\theta}\ddot{f} - 2\hat{H}\dot{f}' + (\theta-1)f'' - \frac{\theta-1}{\theta}g(t)\hat{H}f' = C(t), \tag{6.23}$$

where $C(t)$ is a function of time appearing as a result of the spatial integration symbolized by $\int \mathrm{d}x$. As previously, $C(t)$ can be found by averaging (6.23) along the front. Equation (6.23) is known to be the key ingredient for understanding the parametric flame response to oscillating $g(t)$s (Markstein 1951; Searby & Rochwerger 1991), where inertia (hence, memory) effects play a central role.

6.4. *Steady flame propagation*

In the case of stationary flames, the M-function is time-independent, whence the τ-integration in (5.5) is trivial. One has

$$\frac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}e_k\frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau M(\tilde{x}, t-\tau) = -\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\frac{M(\tilde{x})}{v_+}e_k\frac{\partial}{\partial x_k}\sqrt{r^2 - \frac{(\boldsymbol{r}\cdot\boldsymbol{v}_+)^2}{v_+^2}},$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}M(\tilde{x})\frac{e_k\beta_k}{v_+},$$

where

$$\beta_k = \left(\frac{r_k}{r} - \Omega \frac{v_{k+}}{v_+}\right) \frac{1}{\sqrt{1-\Omega^2}}. \tag{6.24}$$

This vector satisfies

$$\beta_k \beta_k = 1, \qquad \beta_k v_{k+} = 0,$$

i.e. $\boldsymbol{\beta}$ is the unit vector orthogonal to $\boldsymbol{v}_+$. In addition, $\boldsymbol{\beta}$ changes sign at the point $\tilde{x}$ satisfying (4.2). Hence, its scalar product with the complex vector e_i can be written on-shell as

$$e_k \beta_k = -\frac{\omega_+}{v_+} \chi(x - \tilde{x}).$$

Differentiation with respect to x using the formula $\chi'(x) = 2\delta(x)$ makes the $\tilde{x}$-integration trivial, so that (5.5) takes the form

$$2\,(\omega_-)' + \left(1 + i\hat{\mathscr{H}}\right)\left\{[\omega]' - \frac{N v_+^n \sigma_+ \omega_+}{v_+^2}\right\} = 0, \tag{6.25}$$

which is exactly equation (42) of Kazakov (2005*b*).

In connection with (6.25) it is now possible to see why viscous effects have virtually no influence on the shape and the speed of steady flames, except possibly a small effect on the cutoff wavelength (λ_c) itself, the only internal reference length of our problem. The Reynolds number based on λ_c and on the *burnt*-gas properties and speeds is normally about 10^2. The viscous length λ_{vis} – the distance at which the vorticity present behind the front-crests of transverse size $\sim \lambda_c$ dissipates noticeably – thus exceeds λ_c by two orders of magnitude. This is definitely too late (in terms of the Lagrangian time τ) to modify (6.25), whose validity requires only that $\sigma \to \sigma_+$ in an infinitesimal layer ($\tau \to 0^+$) downstream from the front ($\tau = 0$). Concerning unsteady flames, the picture is more complicated, even though the potential flow of fresh gas remains unaffected directly by viscous effects. The problem certainly deserves further study, for one cannot *a priori* exclude that viscosity-related large-scale phenomena occurring far from the front could nonetheless indirectly influence its dynamics. We only wish to mention here that spontaneous flame dynamics is initially little affected, because the Darrieus–Landau time t_{DL} (see § 5.1) is considerably shorter than the viscous time, $t_{vis} = \lambda_{vis}/\theta$, the decay of $M(\tilde{x}, t-\tau)$ for $\tau \gg t_{DL}$ is thus fully controlled by the growth of wrinkle itself.

7. Stability analysis of general steady flame patterns

Let us use (5.5) to derive an equation governing the propagation of small disturbances along a given steady flame pattern. As in the Darrieus–Landau or stationary problems considered above, (5.5) is greatly simplified, because time non-locality is no longer a complication. In fact, since there is no external time-varying field (the stationary regime is assumed to exist) and the flame disturbance is small, it is sufficient to consider perturbations having the form

$$\delta f(x,t) = \tilde{f}(x)\mathrm{e}^{\nu t}, \quad \delta w_-(x,t) = \tilde{w}(x)\mathrm{e}^{\nu t}, \quad \delta u_-(x,t) = \tilde{u}(x)\mathrm{e}^{\nu t},$$

in which case the time dependence is prescribed, and τ-integration in (5.5) is readily performed. We gather the functions $f(x,t)$, $w_-(x,t)$, and $u_-(x,t)$ into an array $\{\xi_\alpha(x,t)\}$, $\alpha = 1, 2, 3$:

$$(\xi_1, \xi_2, \xi_3) = (f, w_-, u_-),$$

and denote by $\xi^{(0)}_\alpha(x)$ the given solution of (6.25). We need to linearize the left-hand side of (5.5) with respect to $\delta\xi(x,t) \equiv \tilde{\xi}(x)\mathrm{e}^{\nu t} = (\tilde{f}, \tilde{w}, \tilde{u})\mathrm{e}^{\nu t}$. Depending on whether the function $M(\tilde{x},t)$ is varied, one finds two types of contributions. If $\tilde{\xi}_\alpha$ comes from terms other than M, then $M(\tilde{x},t)$ is evaluated for $\xi = \xi^{(0)}$ and is hence time-independent. As we saw in the preceding section, (5.5) simplifies in this case to (6.25), so the corresponding contribution to the variation is given by the left-hand side of (6.25) linearized around $\xi^{(0)}$, with M kept fixed. The other contribution comes from variation of $M(\tilde{x},t)$ and has the form

$$\Delta_M = \left(1 + \mathrm{i}\hat{\mathscr{H}}\right)\left\{-\frac{\mathrm{i}}{4}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\, e_k \frac{\partial}{\partial x_k}\int_{\tau_-}^{\tau_+} \mathrm{d}\tau\, \hat{M}_\alpha(\tilde{x})\tilde{\xi}_\alpha(\tilde{x})\mathrm{e}^{\nu(t-\tau)}\right\}',$$

where $\hat{M}_\alpha(\tilde{x})$ is the differential operator obtained by linearizing the function $M(\tilde{x},t)$ around the stationary solution, and setting $\partial/\partial t \to \nu$ afterwards. A straightforward calculation gives

$$\Delta_M = -\frac{\mathrm{e}^{\nu t}}{2}(1 + \mathrm{i}\hat{\mathscr{H}})\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\, \hat{M}_\alpha(\tilde{x})\tilde{\xi}_\alpha(\tilde{x})\frac{\omega_+}{v_+^2}\left\{\exp\left(-\frac{\nu r}{v_+}\mathrm{e}^{-\mathrm{i}\phi}\right)\chi(x-\tilde{x})\right\}',$$

where $\phi \in [-\pi, +\pi]$ is the angle between the vectors $\boldsymbol{r}$, $\boldsymbol{v}_+$, defined positive if the rotation from $\boldsymbol{v}_+$ to $\boldsymbol{r}$ is clockwise. It is not difficult to verify that the x-differentiation of the step function in the above expression gives rise to a term that is just the variation of the left-hand side of (6.25) under a variation of the function $M(\tilde{x},t)$. Furthermore, one has

$$(r\sin\phi)' = \tau_i \frac{\partial(r\sin\phi)}{\partial x_i} = \tau_i\beta_i\chi(x-\tilde{x}),$$

where $\tau_i = \varepsilon_{ik}n_k$ is the unit vector tangential to the flame front. Taking also into account that

$$\varepsilon_{ik}\beta_i = \frac{v_{k+}}{v_+}\chi(x-\tilde{x}),$$

we find

$$(r\sin\phi)' = \varepsilon_{ik}n_k\beta_i\chi(x-\tilde{x}) = \frac{v_{k+}n_k}{v_+} \equiv \frac{v_+^n}{v_+}.$$

Similarly,

$$(r\cos\phi)' = \frac{v_{k+}\tau_k}{v_+} \equiv \frac{v_+^\tau}{v_+}.$$

Putting all these results together and omitting the factor $\mathrm{e}^{\nu t}$, we obtain the following equation for the x-dependent parts of the perturbations

$$2\tilde{\omega}' + \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\,\tilde{\xi}_\alpha(\tilde{x})\frac{\delta}{\delta\xi_\alpha(\tilde{x})}(1 + \mathrm{i}\hat{\mathscr{H}})\left\{[\omega]' - \frac{N v_+^n\sigma_+\omega_+}{v_+^2}\right\}$$
$$= \frac{\mathrm{i}\nu}{2}(1 + \mathrm{i}\hat{\mathscr{H}})\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\,\hat{M}_\alpha(\tilde{x})\tilde{\xi}_\alpha(\tilde{x})\frac{(v_+^n + \mathrm{i}v_+^\tau)\omega_+}{v_+^4}\exp\left(-\frac{\nu r}{v_+}\mathrm{e}^{-\mathrm{i}\phi}\right)\chi(x-\tilde{x}), \quad (7.1)$$

where $\delta/\delta\xi_\alpha(\tilde{x})$ denotes functional differentiation, and $\tilde{\omega} = \tilde{u} + \mathrm{i}\tilde{w}$. Together with the linearized evolution equation, (7.1) can be used to perform the stability analysis of general steady flame configurations, given by the solutions of (5.8), (6.25). To the best of our knowledge, no closed equation such as (7.1) has previously been derived to address this problem. Its applications will be presented elsewhere.

8. Discussion and conclusions

The results presented in this paper solve the problem of non-perturbative description of an unsteady premixed flame propagation with arbitrary gas expansion. Supplemented by the evolution equation, (5.5) gives a closed description of unsteady flame propagation in its most general form. Thus, as in the stationary two-dimensional case, the dilemma mentioned in the introduction is resolved with a negative response for two-dimensional unsteady flame propagation. The conclusion that the detailed structure of the bulk gas flow is actually unnecessary to describe front dynamics is even more striking in the latter case. Indeed, the highly complicated rotational flow downstream is continuously changing in time, and is naturally expected to have an exceedingly complicated non-local influence on the flame-front evolution. We proved, however, that all necessary information about this influence is encoded in the complex history of the combination $M = N\bar{v}_+^n\sigma_+$. In this connection, it is worth mentioning a curious circumstance. It was seen in § 4 that the rotational component depends on the spatial coordinates through the complex combinations $\tau_\pm$ appearing as the limits of integration in the complex-time plane. It is easy to see that for any curved flame configuration, there are always points at the front where $\Omega < 0$. For such points, the time argument of the function $M(\tilde{x}, t - \tau)$ in the integrand of (4.11) has a real part $> t$. In other words, integration over such points is in a sense looking into the flame's future. This does not lead to any conflict with causality, because the corresponding contribution is eventually annihilated by the operator $(1 + \mathrm{i}\hat{\mathscr{H}})$ in (5.5). Indeed, the rotational part of $\boldsymbol{v}^v$ comes from integration over $\tilde{x}$ such that the vectors $\boldsymbol{r}$ and $\bar{\boldsymbol{v}}_+$ are almost parallel, i.e. $\Omega \approx 1$, and therefore, the contribution of points with $\Omega < 0$ is a pure potential satisfying (5.1). However, it is necessary to retain this contribution in the intermediate formulas so as to guarantee continuity of the potential component.

The paradox of the seemingly 'teleological' structure of (5.5) is closely related to the analyticity properties of $M(x, t)$ in the complex t-plane, which allowed us to simplify (4.4) to (4.8). Our considerations were put forward under the very weak assumption of exponential boundedness of this function, which is certainly sufficient for the investigation of any flame propagation phenomena. In particular, the linear stability problem analysed in § 6.1 provides an example of $M(x, t)$ which is analytic in the complex plane, so that the question as to whether the contour deformation is legitimate does not arise at all. In this simplest case knowing the exponentials $\exp(\nu t)$ in (6.3) at current time t allows one to predict their future, so that the 'teleological' question does not arise either. Matters become more subtle, however, in the case of flame propagation in a time-dependent gravitational field. Suppose that an experimentalist plans to let the burner fall freely at some time instant t_0. This means that $g(t)$, and hence, $M(\tilde{x}, t)$ (cf. (3.6), (6.20)) will have singularities at $\tau_0 = t_0 \pm \mathrm{i}\Delta t$, where Δt is of the order of the time taken to switch off the field. As is shown in the Appendix, crossing these singularities by the contour $C_- \cup C_+$ in figure 2 gives rise to a potential contribution that satisfies conditions (a)–(c) of § 4, and hence does not change (5.5), thereby resolving the causality issue.

To conclude, (5.5) opens a wealth of key developments in theoretical and numerical combustion (extended propagation laws, coupling with acoustics, burners, etc.), not to mention the other fronts evoked at the beginning of this paper. In particular, (7.1) allows a direct analytical investigation of small disturbances propagating on steady front patterns such as Bunsen- or V-flames in two-dimensional configurations. The extension of the above results to three-dimensional problems remains an open question: one of the main difficulties is the generalization of the $(1+\mathrm{i}\hat{\mathscr{H}})$ operator used to project out the potential contributions of the burnt-gas flow.

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Appendix

When deriving expression (4.11) for the rotational component in §4 we omitted the contribution of the contour integral in (4.7), retaining only the singular contribution of the poles $\tau_{\pm}$. That this operation respects property (c) has already been shown in §4.1. We will now prove that (4.11) indeed correctly reproduces the near-the-front vorticity distribution of the burnt-gas flow. In essence, the following calculation reproduces the consistency check given in appendix A of (Kazakov 2005*b*). First of all, taking into account the formula

$$\frac{\partial \tau_{\pm}}{\partial x_i} = \pm\frac{\mathrm{i}}{\bar{v}_+}\left(\beta_i \mp \mathrm{i}\frac{\bar{v}_{i+}}{\bar{v}_+}\right), \tag{A 1}$$

which is readily verified using the definitions (4.5), (6.24), one has

$$\begin{aligned}\varepsilon_{ki}\partial_k v_i^v &= \frac{1}{4}\frac{\partial}{\partial x_i}\left\{\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\frac{\mathrm{e}^{-\mu r}}{\bar{v}_+}\left[M(\tilde{x}, t-\tau_+)\left(\beta_i - \mathrm{i}\frac{\bar{v}_{i+}}{\bar{v}_+}\right)\right.\right.\\ &\qquad \left.\left.+ M(\tilde{x}, t-\tau_-)\left(\beta_i + \mathrm{i}\frac{\bar{v}_{i+}}{\bar{v}_+}\right)\right]\right\}_{\mu=0^+}\\ &= \frac{1}{2}\frac{\partial}{\partial x_i}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\frac{\mathrm{e}^{-\mu r}}{\bar{v}_+}M(\tilde{x}, t-\tau_+)\left(\beta_i - \mathrm{i}\frac{\bar{v}_{i+}}{\bar{v}_+}\right)\right\}_{\mu=0^+}.\end{aligned}$$

Following the argument given in §4, we differentiate with respect to x_i under the sign of the $\tilde{x}$-integral, and see that the derivative of $\mathrm{e}^{-\mu r}$ leads to an integral proportional to μ. Therefore, the only non-zero term contributed by this integral after μ is continued to zero is an $\boldsymbol{x}$-independent constant that does not contribute to $\mathrm{rotD}\boldsymbol{v}^v$. Next, it is easily verified that

$$\left(\beta_k - \mathrm{i}\frac{\bar{v}_{k+}}{\bar{v}_+}\right)^2 \equiv 0, \tag{A 2}$$

so differentiation of $M(\tilde{x}, t-\tau_+)$ likewise gives zero. One may also note that $\boldsymbol{\beta}$ can be written as

$$\beta_i = \frac{\varepsilon_{ik}\bar{v}_{k+}}{\bar{v}_+}\chi(\varepsilon_{lm}r_l\bar{v}_{m+}), \tag{A 3}$$

since it is orthogonal to $\bar{\boldsymbol{v}}_+$ and changes sign at the point satisfying (4.2). Thus, we find

$$\begin{aligned}\varepsilon_{ki}\partial_k v_i^v &= \mathrm{Re}\left\{\int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\frac{\mathrm{e}^{-\mu r}}{\bar{v}_+^2}M(\tilde{x}, t-\tau_+)\varepsilon_{ik}\bar{v}_{k+}\varepsilon_{in}\bar{v}_{n+}\delta(\varepsilon_{lm}r_l\bar{v}_{m+})\right\}_{\mu=0^+}\\ &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{x}\, M(\tilde{x}, t-r/\bar{v}_+)\delta(\varepsilon_{lm}r_l\bar{v}_{m+}).\end{aligned} \tag{A 4}$$

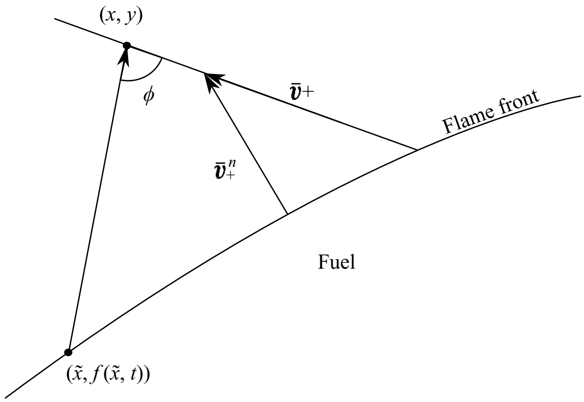

FIGURE 3. Near-the-front structure of the flow downstream.

The factor $e^{-\mu r}$ and the symbol of analytic continuation have been omitted in the last expression because it is explicitly finite. The argument of the δ-function is zero when the vectors r_i and $\bar{v}_{i+}$ are parallel. Near this point, one has approximately

$$\varepsilon_{lm} r_l \bar{v}_{m+} = r\bar{v}_+\phi.$$

Moreover, a simple geometric consideration (see figure 3) shows that $\mathrm{d}\tilde{x}$ near the same point can be written as

$$\mathrm{d}\tilde{x} = -\frac{r\bar{v}_+ \mathrm{d}\phi}{N\bar{v}^n_+},$$

where all quantities are taken at the time instant t. Substituting these expressions into (A 4), and taking into account the relation

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

yields

$$\varepsilon_{ki}\partial_k v^v_i = \int \mathrm{d}\phi\,\delta(\phi)\frac{M(\tilde{x}, t - r/\bar{v}_+)}{N(\tilde{x}, t)\bar{v}^n_+(\tilde{x}, t)}. \tag{A 5}$$

If the observation point is taken at the flame front, r goes to zero together with ϕ. Also recalling the definition (3.6) of the function $M(\tilde{x}, t)$, we finally arrive at the desired identity

$$\left(\varepsilon_{ki}\partial_k v^v_i\right)_+ = \sigma_+(\tilde{x}, t).$$

Let us now return to the question of whether one can perform the contour deformation in the complex τ-plane, used in the derivation of (4.9). We note, first of all, that this question pertains wholly to the structure of the potential component of the gas velocity. Indeed, under our general assumption of the existence of a short-wavelength cutoff, all functions involved are smooth functions of time (because any structure of finite size takes finite time to develop). Hence, these functions (in particular, the function $M(\tilde{x}, t)$) are analytic in a finite strip about the real axis in the complex t-plane. On the other hand, we also know that the value of vorticity at any given point (x, y) near the front is equal to its value at the point on the front, satisfying $\Omega = 1$, in which case the poles $\tau_\pm$ take on the real value $r/\bar{v}_+$. For $\tilde{x}$ in the vicinity of that point, $\tau_\pm$ belong to the analyticity domain of $M(\tilde{x}, t)$, and hence, of the function $\mathscr{M}(\tilde{x}, \tau, t)$. Thus, use of the Cauchy theorem and the contour deformation performed in §4 are legitimate for these $\tilde{x}$, while integration over all other $\tilde{x}$ gives rise to a potential contribution. This result, however, is not a concluding consideration, since the arguments just given prove the potentiality of this contribution in the neighbourhood of a given observation point. To prove it for all $x \in [0, 1]$, the contour $C_- \cup C_+$ in figure 2 should be moved to the left of $\tau_\pm$ for all $\tilde{x}$. Let us show that this is indeed possible under the assumption already used to derive (4.11) namely, that $M(\tilde{x}, t)$ considered as the function of complex t is exponentially bounded near $t = \infty$. Suppose, for instance, that this function is meromorphic, i.e. it only has poles, of arbitrary order, in the t-plane. Any pole of order n in the function $M(\tilde{x}, t)$ becomes an $(n-1)$th-order pole with respect to τ in $\mathscr{M}(\tilde{x}, \tau, t)$. Hence, crossing these poles by $C_- \cup C_+$ does not change the right-hand side of (4.7) unless $n = 2$ or $n = 1$. Consider first the case $n = 2$. Then $\mathscr{M}(\tilde{x}, \tau, t)$ has simple poles at some point τ_0 and its complex conjugate τ_0^* (τ_0 may depend on $\tilde{x}, t$, but, for brevity, we do not write this dependence explicitly). We have to show that their contribution to the right-hand side of (4.7), given by the Cauchy theorem as ('c.c.' stands for complex conjugate)

$$\frac{2\pi\mathrm{i}}{4\pi} \operatorname{res} \mathscr{M}(\tilde{x}, \tau_0, t) \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} + \text{c.c.}, \tag{A 6}$$

gives rise to a field V_i that satisfies conditions (a)–(c). Substituting this expression into (4.1) yields (we do not introduce intermediate regularization because the $\tilde{x}$-integral will be shown to converge)

$$V_i = -\frac{\mathrm{i}}{4} \varepsilon_{ik} \int_{-A}^{+A} \mathrm{d}\tilde{x} \operatorname{res} \mathscr{M}(\tilde{x}, \tau_0, t) \partial_k \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} + \text{c.c.} \tag{A 7}$$

Property (a) is evidently satisfied. To prove (b) we write, using (A 1), (A 2) and (A 3),

$$\begin{aligned} &\partial_k \int_{-A}^{+A} \mathrm{d}\tilde{x} \operatorname{res} \mathscr{M}(\tilde{x}, \tau_0, t) \partial_k \left\{ \frac{1}{\tau_0 - \tau_+} + \frac{1}{\tau_0 - \tau_-} \right\} \\ &= \int_{-A}^{+A} \mathrm{d}\tilde{x} \operatorname{res} \mathscr{M}(\tilde{x}, \tau_0, t) \left\{ \frac{\partial_k^2 \tau_+}{(\tau_0 - \tau_+)^2} + \frac{\partial_k^2 \tau_-}{(\tau_0 - \tau_-)^2} \right\} \\ &= 2\mathrm{i} \int_{-A}^{+A} \mathrm{d}\tilde{x} \operatorname{res} \mathscr{M}(\tilde{x}, \tau_0, t) \delta(\varepsilon_{lm} r_l \bar{v}_{m+}) \left\{ \frac{1}{(\tau_0 - \tau_+)^2} - \frac{1}{(\tau_0 - \tau_-)^2} \right\} = 0, \end{aligned}$$

since $\tau_+ = \tau_-$ when the argument of the δ function is zero. Thus, $\operatorname{rot} \boldsymbol{V} = 0$. Lastly, $\mathscr{M}(\tilde{x}, \tau, t)$ is periodic in $\tilde{x}$, and therefore, so is its pole. Taking also into account that $\tau_\pm = O(|\tilde{x}|)$, $\partial_i \tau_\pm = O(1)$ for $|\tilde{x}| \to \infty$, one sees that the $\tilde{x}$-integral in (A 7) is convergent in the limit $A \to \infty$ for all $\boldsymbol{x}$.

In the case $n = 1$ the function $\mathscr{M}(\tilde{x}, \tau, t)$ contains a logarithmic singularity of the form $a(\tilde{x}, t) \ln(\tau - \tau_0)$ (and its complex conjugate). Crossing this singularity leads to the 2π jump in $\arg(\ln(\cdot))$ for all points of the contour $C_- \cup C_+$, located at one side of the point τ_0. Hence, expression (A 6) is replaced in this case by the following

$$\frac{2\pi\mathrm{i} a(\tilde{x}, t)}{4\pi} \int_0^{\tau_0} \mathrm{d}\tau \left\{ \frac{1}{\tau - \tau_+} + \frac{1}{\tau - \tau_-} \right\} + \text{c.c.}$$

The proof of properties (a)–(c) is exactly the same as before.

Let us finally consider the case when the function $M(\tilde{x}, t)$ has branch singularities. If these singularities are connected by a number of cuts so that $M(\tilde{x}, t)$ is meromorphic in the cut τ-plane, then so is the function $\mathscr{M}(\tilde{x}, \tau, t)$, and moving the contour of integration beyond a cut results in a contribution to the right-hand side of (4.7) of the form

$$\frac{1}{4\pi}\int_{C_0} \mathrm{d}\tau\,[\mathscr{M}](\tilde{x},\tau,t)\left\{\frac{1}{\tau-\tau_+}+\frac{1}{\tau-\tau_-}\right\}+\text{c.c.},$$

where $[\mathscr{M}](\tilde{x},\tau,t)$ denotes the jump of the function $\mathscr{M}(\tilde{x},\tau,t)$ across the cut C_0 (if $\mathscr{M}(\tilde{x},\tau,t)$ is singular at C_0, the above integral can be replaced by the integral of this function over a closed contour embracing the cut). If this cut has finite length, then the above considerations once again literally apply. However, in the case of an infinite cut, the τ-integral is apparently divergent. This means that such cuts, if any, are allowed only in regions where $\mathscr{M}(\tilde{x},\tau,t)$ satisfies more restrictive conditions than the exponential boundedness. We will not pursue any details here, because the physical significance of such cuts is not clear.

Thus, the function $M(\tilde{x},t)$ is allowed to have any number of branch singularities in the complex t-plane, connected by cuts of finite length, as well as any number of poles of arbitrary order to justify the contour deformation used in § 4.

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